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STATISTICAL PROPERTIES OF INHOMOGENEOUS SOLID MEDIA: CENTRAL MOMENT FUNCTIONS OF MATERIAL CHARACTERISTICS

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Statistical properties of multicomponent inhomogeneous media are investigated. A general definition of the concentration and coordinate functional structure of central moment functions of material characteristics is presented. It is established that such functional structure is highly sensitive to differences in statistical properties of components. The class of symmetric media for which central moment functions are greatly simplified are considered.

1. The term material characteristics, as used here, comprises quantities of the type of tensors of the elasticity modulus, of permittivity and, also, scalars such as density, specific heat, etc. For simplicity of presentation tensor indices of material characteristics are almost everywhere omitted.

Let there be a random tensor field $\lambda(\mathbf{r})$ (for instance, of elasticity moduli). Extending the characteristic functions derived in [1, 2] for two-component media to the case of arbitrary number of components, we obtain

$$f_i(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in U_i, \quad i \neq 1 \\ 0, & \mathbf{r} \notin U_i, \quad \sum_{i=1}^{N} U_i = U \end{cases} \tag{1.1}$$

where U_i is the region occupied by the i-th component and N is the number of components. The characteristic function f_1 for the first component is determined by the obvious equality

$$\sum_{i=1}^{N} f_i(\mathbf{r}) = 1 \tag{1.2}$$

The separation of the first component is not due here to any specific reason. We shall, however, always have in view the matrix medium in which a particular part is played by at least one component-matrix, as the simplest example of such medium.

Using (1.1) we define the random field $\lambda(\mathbf{r})$ by the formula

$$\lambda(\mathbf{r}) = \lambda_1 + \sum_{i=2}^{N} \Delta\lambda_i f_i(\mathbf{r}) \equiv \lambda_1 + \Delta\lambda_i f_i(\mathbf{r}), \quad \Delta\lambda_i \equiv \lambda_i - \lambda_1 \tag{1.3}$$

where λ_i is the value of λ for the i-th component.

The fluctuation λ'' of field λ is

$$\lambda''(\mathbf{r}) = \Delta\lambda_i f_i''(\mathbf{r}), \quad x''(\mathbf{r}) \equiv x(\mathbf{r}) - \langle x(\mathbf{r})\rangle \tag{1.4}$$

where angle brackets denote statistical averaging.

The random field $\lambda(\mathbf{r})$ may be specified either by moment or central moment functions of various orders. The moment function of order ω of field λ is by definition of the form [3, 4]

$$M_\omega(\mathbf{R}_n) = \langle[\lambda(\mathbf{r}_1)\otimes]^{\omega_1}\cdots[\lambda(\mathbf{r}_n)\otimes]^{\omega_n}\rangle, \quad \mathbf{R}_n \equiv (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \quad (1,5)$$

For the central moment function of order ω we correspondingly have

$$\mu_\omega(\mathbf{R}_n) = \langle[\lambda''(\mathbf{r}_1)\otimes]^{\omega_1}\cdots[\lambda''(\mathbf{r}_n)\otimes]^{\omega_n}\rangle, \quad \omega = \sum_{a=1}^{n}\omega_a \qquad (1.6)$$

In (1.5) and (1.6) the symbol $\otimes$ denotes a direct tensor product.

It is readily seen [5] that the use of characteristic functions f_i makes it possible to reduce the calculation of M_ω to the determination of function

$$P^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = \langle f_i(\mathbf{r}_1^i) f_j(\mathbf{r}_2^j)\ldots f_k(\mathbf{r}_n^k)\rangle, \; \mathbf{R}_n^{ij\ldots k} \equiv (\mathbf{r}_1^i, \mathbf{r}_2^j, \ldots, \mathbf{r}_n^k) \quad (1.7)$$

which has the meaning of probability that points $\mathbf{r}_1^i$, $\mathbf{r}_2^j$, ..., $\mathbf{r}_n^k$ belong, respectively, to regions U_i, U_j, ..., U_k, where $i, j, \ldots, k \neq 1$. The additional superscript at the radius vector indicates the region to which the respective point belongs. The introduced function was similarly investigated in [2] in the case of two-component medium.

The probability $P^n_{ij\ldots k}$ is invariant with respect to the simultaneous permutation of coordinates of points and of indices. Furthermore they have the following properties:

$$\sum_{i=1}^{N} P^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = P^{n-1}_{j\ldots k}(\mathbf{R}_{n-1}^{j\ldots k}) \qquad (1.8)$$

$$P^n_{ij\ldots l\, l\ldots l}(\mathbf{R}_q^{ij\ldots l}, \mathbf{r}, \ldots, \mathbf{r}) = P^q_{ij\ldots l}(\mathbf{R}_q^{ij\ldots l}), \quad \mathbf{r} \equiv \mathbf{r}_q^{\,l}$$

$$P^n_{ij\ldots l\, m\ldots k}(\mathbf{R}_q^{ij\ldots l} \,|\, \overline{\mathbf{R}}_{n-q}^{m\ldots k}) = P^q_{ij\ldots l}(\mathbf{R}_q^{ij\ldots l})\, P^{n-q}_{m\ldots k}(\overline{\mathbf{R}}_{n-q}^{m\ldots k})$$

$$\overline{\mathbf{R}}_{n-q}^{m\ldots k} \equiv (\mathbf{r}_{q+1}^m, \ldots, \mathbf{r}_n^k)$$

where the vertical stroke separates the groups of points $\mathbf{R}_q^{ij\ldots l}$ and $\overline{\mathbf{R}}_{n-q}^{m\ldots k}$ which are at infinitely great distances from each other; the total number of indices $ij\ldots l$ is q, and the number of indices $l\ldots l$ and $m\ldots k$, and also the number of variables $\mathbf{r}, \ldots, \mathbf{r}$ is $(n-q)$.

Below we assume that the field $f_i(\mathbf{r})$ is statistically homogeneous and isotropic, because of which functions $P^n_{ij\ldots k}$ are invariant with respect to translations and rotations. Hence

$$P_i^1(\mathbf{r}_1^i) = v_i, \quad v_i = V_i / V, \quad V = \sum_{i=1}^{N} V_i \qquad (1.9)$$

where v_i is the volume concentration of the i-th component.

Using (1.8) we similarly represent P_{ij}^2 in the form

$$P_{ij}^2(\mathbf{r}_1^i, \mathbf{r}_2^j) = v_i\delta_{ij}\varphi_0^2(\mathbf{r}_1^i, \mathbf{r}_2^j) + v_i v_j \varphi_1^2(\mathbf{r}_1^i, \mathbf{r}_2^j), \quad i, j \neq 1 \qquad (1.10)$$

where, as in (1.8) summation is not carried out with respect to the twice occuring indices, and functions φ_q^2 satisfy conditions

$$\varphi_0^2(\mathbf{r}_1^i, \mathbf{r}_1^i) = \varphi_1^2(\mathbf{r}_1^i \,|\, \mathbf{r}_2^j) = 1, \quad \varphi_0^2(\mathbf{r}_1^i \,|\, \mathbf{r}_2^j) = \varphi_1^2(\mathbf{r}_1^i, \mathbf{r}_1^i) = 0 \quad (1.11)$$

If certain constraints are imposed on field $f_i(\mathbf{r})$, functions φ_q^2 can be obtained in explicit form [5].

Extending (1.10) to the case of arbitrary n in the notation of (1.8) we have

$$P^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = \sum_{\sigma_n}\sum_{q=0}^{n-1} v^q_{ij\ldots k}\varphi_q{}^n(\mathbf{R}_q^{ij\ldots l}; \overline{\mathbf{R}}^{m\ldots k}_{n-q}) \tag{1.12}$$

$$P^\circ = 1, \quad i, j\ldots, k \neq 1$$

$$v^q_{ij\ldots k} = v_i v_j \ldots v_l \sum_{p=2}^{N} v_p \delta_{mp}\ldots\delta_{kp}, \quad \mathbf{R}_n^{ij\ldots k} = (R_q^{ij\ldots l}, \overline{\mathbf{R}}^{m\ldots k}_{n-q})$$

where functions $\varphi_q{}^n$ have q separated points, and summation is carried out over all nonidentical permutations of σ_n, i.e. of the first n (in this case of all) points with simultaneous permutation of respective indices. The asymptotic formulas for these follow form (1.8) and are of a form similar to (1.11).

Note that function φ^n_{n-1} does not contain the separated point.

In the opposite case, taking into account the total symmetry of $v^n_{ij\ldots k} = v_i v_j \ldots v_k$, they are symmetrized owing to the action of σ_n.

As an example, we adduce the formula for the three-point probability

$$\begin{aligned} P^3_{ijk}(\mathbf{r}_1{}^i, \mathbf{r}_2{}^j, \mathbf{r}_3{}^k) = {} & \varphi_0{}^3(\mathbf{r}_1{}^i, \mathbf{r}_2{}^j, \mathbf{r}_3{}^k)\sum_{p=2}^{N} v_p\delta_{ip}\delta_{jp}\delta_{kp} + \\ & \varphi_1{}^3(\mathbf{r}_1{}^i; \mathbf{r}_2{}^j, \mathbf{r}_3{}^k)\, v_i \sum_{p=2}^{N} v_p\delta_{jp}\delta_{kp} + \\ & \varphi_1{}^3(\mathbf{r}_2{}^j; \mathbf{r}_3{}^k, \mathbf{r}_1{}^i)\, v_j \sum_{p=2}^{N} v_p\delta_{ip}\delta_{pp} + \\ & \varphi_1{}^3(\mathbf{r}_3{}^k; \mathbf{r}_1{}^i, \mathbf{r}_2{}^j)\, v_k \sum_{p=2}^{N} v_p\delta_{ip}\delta_{jp} + v_i v_j v_k \varphi_2{}^3(\mathbf{r}_1{}^i, \mathbf{r}_2{}^j, \mathbf{r}_3{}^k) \end{aligned} \tag{1.13}$$

Functions $\varphi_q{}^n$ introduced in (1.12) can be associated with various probabilities. Thus, for instance, the probability that point $\mathbf{r}_1{}^i$ belongs to region U_2, while points $\mathbf{r}_2{}^j$ and $\mathbf{r}_3{}^k$ belong to region U_3, according to (1.13), is

$$P^3_{233}(\mathbf{R}_3) = v_2 v_3 \varphi_1{}^3(\mathbf{r}_1; \mathbf{r}_2, \mathbf{r}_3) + v_2 v_3 v_3 \varphi_2{}^3(\mathbf{R}_3) \tag{1.14}$$

and simultaneously function $\varphi_2{}^3$ determines the probability that all three points belong to different regions

$$P^3_{234}(\mathbf{R}_3) = v_2 v_3 v_4 \varphi_2{}^3(\mathbf{R}_3) \tag{1.15}$$

In addition to probability $P^n_{ij\ldots k}$, which has the meaning of the n-th order moment of field f_i, it is expedient to introduce in the analysis its central moment

$$T^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = \langle f_i''(r_1{}^i) f_j''(\mathbf{r}_2{}^j)\ldots f_k''(\mathbf{r}_n{}^k)\rangle \tag{1.16}$$

related to $P^n_{ij\ldots k}$ by the formula

$$T^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = \sum_{\sigma_n}\sum_{q=0}^{n}(-1)^q v_i v_j \ldots v_l P^{n-q}_{m\ldots k}(\overline{\mathbf{R}}^{m\ldots k}_{n-q}) \tag{1.17}$$

The summation in (1.17) over permutations of n points results in the appearance of $N_k = n!\, /\, [k!\,(n-k)!]$ terms of the form $v_i v_j \ldots v_l P^{n-q}_{m\ldots k}$ whose dependence on coordinates is determined by the respective set of $(n-q)$ points from the full set of n points.

The calculation of central moments μ_ω reduces to the determination of functions $T^n_{ij\ldots k}$.

2. We transform functions $P^n_{ij\ldots k}$ and $T^n_{ij\ldots k}$ so as to obtain their explicit expressions in terms of dispersion $D^n_{ij\ldots k}$ of the characteristic function f_i. By dispersion $D^n_{ij\ldots k}$ we understand the value of functions $T^n_{ij\ldots k}$ when $\mathbf{r}_1{}^i = \mathbf{r}_2{}^j = \ldots = \mathbf{r}_n{}^k$, i.e. when $\mathbf{R}_n^{ij\ldots k} = 0$.

Taking into account that according to (1.7) or (1.12)

$$P^n_{ij\ldots k}(0) = \sum_{p=2}^{N} v_p \delta_{ip}\delta_{jp}\ldots\delta_{kp} \tag{2.1}$$

from (1.17) with allowance for definition (1.12) we obtain

$$D^n_{ij\ldots k} \equiv T^n_{ij\ldots k}(0) = \sum_{\sigma_n}\sum_{q=0}^{n}(-1)^q v^q_{ij\ldots k} \tag{2.2}$$

In what follows we shall need functions

$$\begin{aligned}
&\pi_q{}^n(\mathbf{R}_q;\overline{\mathbf{R}}_{n-q}) = \sum_{\sigma_q}\sum_{t=0}^{q}\varphi_t{}^n(\mathbf{R}_t;\overline{\mathbf{R}}_{n-t})\\
&\pi_0{}^\circ = \pi_1{}^1 = 1, \quad \pi^n_{n-1} = 0\\
&\tau_q{}^n(\mathbf{R}_q;\overline{\mathbf{R}}_{n-q}) = \sum_{\sigma_q}\sum_{t=0}^{q}(-1)^t\pi_t^{n-q+t}(\mathbf{r}_{q-t+1},\ldots,\mathbf{r}_q;\overline{\mathbf{R}}_{n-q})\\
&\tau_0{}^\circ = \tau_1{}^1 = \tau^n_{n-1} = 0
\end{aligned} \tag{2.3}$$

where in order to avoid clumsiness of formulas superscripts at radius vectors are omitted. Functions (2.3) satisfy certain limit conditions implied by (1.8) in the meaning of (1.11).

From this, for instance for $n = 2$, we have

$$\begin{aligned}
&\pi_0{}^2(\mathbf{R}_2) = \tau_0{}^2(\mathbf{R}_2) = \varphi_0{}^2(\mathbf{R}_2), \quad \pi_2{}^2(\mathbf{R}_2) = \varphi_0{}^2(\mathbf{R}_2) + \varphi_1{}^2(\mathbf{R}_2)\\
&\tau_2{}^2(\mathbf{R}_2) = \pi_0{}^\circ - \pi_1{}^1(\mathbf{r}_1) - \pi_1{}^1(\mathbf{r}_2) + \pi_2{}^2(\mathbf{R}_2) = \pi_2{}^2(\mathbf{R}_2) - 1
\end{aligned} \tag{2.4}$$

Using (2.3) functions (1.12) and (1.17) can be reduced to the form

$$\begin{aligned}
&P^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = \sum_{\sigma_n}\sum_{q=0}^{n} v_i v_j\ldots v_l D^{n-q}_{m\ldots k}\pi_q{}^n(\mathbf{R}_q^{ij\ldots l};\overline{\mathbf{R}}_{n-q}^{m\ldots k})\\
&T^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k}) = \sum_{\sigma_n}\sum_{q=0}^{n}(-1)^q v_i v_j\ldots v_l D^{n-q}_{m\ldots k}\tau_q{}^n(\mathbf{R}_q^{ij\ldots l};\overline{\mathbf{R}}_{n-q}^{m\ldots k})
\end{aligned} \tag{2.5}$$

The advantage of formula (2.5) over (1.12) and (1.17) is in that in it dispersions of the characteristic function appear in explicit form, which is particularly convenient for the calculation of μ_ω. Below we consider central moments μ_ω in the case when all $\omega_a = 1$.

Using the definitions (1.4), (1.6), and (1.16) we obtain

$$\mu_{11\ldots 1}(\mathbf{R}_n) \equiv \mu_n(\mathbf{R}_n) = T^n_{ij\ldots k}(\mathbf{R}_n^{ij\ldots k})\,\Delta\lambda_i\otimes\Delta\lambda_j\otimes\ldots\otimes\Delta\lambda_k \tag{2.6}$$

where summation with respect to indices $i, j, \ldots, k$, which denote the component ordinal numbers, is assumed to be from two to N. The substitution into this formula of the expression $T^n_{ij\ldots k}$ in (2.5) yields

$$\mu_n(\mathbf{R}_n) = \sum_{\sigma_n} \sum_{q=0}^{n} (-1)^q v_i v_j \ldots v_l \times D^{n-q}_{m\ldots k} \Delta\lambda_i \otimes \Delta\lambda_j \otimes \ldots \otimes \Delta\lambda_k \tau_q{}^n (\mathbf{R}_q^{ij\ldots l}; \overline{\mathbf{R}}_{n-q}^{m\ldots k}) \quad (2.7)$$

We use (2.7) for defining the second order central moment of the tensor field λ_α, where α is to be understood to represent the totality of tensor indices of field λ. From (2.7) we have

$$\mu_{\alpha\beta}(\mathbf{R}_2) = [v_i v_j \tau_2{}^2 (\mathbf{R}_2{}^{ij}) + D_{ij}{}^2 \tau_0{}^2 (\mathbf{R}_2{}^{ij})] \Delta\lambda_{i\alpha} \Delta\lambda_{j\beta} \quad (2.8)$$

Because summation in formulas (2.7) and (2.8) is carried out with respect to coordinates, the dependence of central moments of field λ on coordinates and tensors is not separated. It should also be noted that functions $\tau_q{}^n$, as well as functions $\varphi_q{}^n$, depend not only on coordinates of points which determine the dependence of μ_n on coordinates but, also, on that to which components belong the points contained in $\tau_q{}^n$. Thus in the general case $\tau_2{}^2 (\mathbf{r}_1{}^i, \mathbf{r}_2{}^j) \neq \mathbf{r}_2{}^2 (\mathbf{r}_1{}^j, \mathbf{r}_2{}^i)$, when $i \neq j$.

The dependence of $\tau_q{}^n$ on component indices may be due to their differences related to the spatial distribution and to the shape of regions occupied by components. Furthermore it may be due to incomplete definition of the dependence on concentration by coefficients $v_i v_j \ldots v_l D^{n-q}_{m\ldots k}$ in (2.5).

Let the considered medium be such that function $\tau_q{}^n$ is symmetric with respect to permutations of indices of components when the position of points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ is fixed. Functions $\tau_q{}^n$ then satisfy the condition

$$\tau_q{}^n (\mathbf{R}_q^{ij\ldots l}; \overline{\mathbf{R}}_{n-q}^{m\ldots k}) = \tau_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q}) \quad (2.9)$$

which we shall call the condition of symmetry. It results in that the differences in statistical properties of all components, except the first, reduce in conformity with (2.5) to their concentration. A similar condition can also be formulated for functions $\pi_q{}^n$.

Condition (2.9) considerably simplifies the expression for the central moment μ_n. The substitution of (2.9) into (2.7) yields

$$\mu_n^{(1)}(\mathbf{R}_n) = \sum_{\sigma_n} \sum_{q=0}^{n} [\lambda_1'' \otimes]^q D^{(\lambda)}_{n-q} \tau_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q}) \quad (2.10)$$

$$(\lambda_1'' = -v_i \Delta\lambda_i, \ D_n^{(\lambda)} \equiv \mu_n(0))$$

where the first of equalities appearing in parentheses follows from (1.4). The superscript at μ_n is shown here for stressing the importance of the first component. A medium that satisfies conditions (2.9) and (2.10) is called below almost symmetric.

The described above determination of function μ_n, obtained with the use of characteristic function (1.1), may also be effected by using functions of the form

$$f_i^{(p)}(\mathbf{r}) = \begin{cases} 1, \ \mathbf{r} \in U_i, & i \neq p \\ 0, \ \mathbf{r} \notin U_i, & \sum_{i=1}^{N} U_i = U \end{cases} \quad (2.11)$$

Function (2.11) defines the same medium as function (1.1), hence the result of calculating μ_n must evidently be the same. However in order to represent μ_n in the form (2.7) it is necessary to bear in mind that functions $\tau_q{}^n$ are, generally speaking, different, $\Delta\lambda_i = \lambda_i - \lambda_p$, and summation with respect to subscripts $i, j, \ldots, k$ excludes the p-th component.

In the case of almost symmetric medium the calculation of μ_n by (1.1) yields (2.10), while the use of (2.11) leads to an expression of the form (2.7). Hence when the medium contains a component whose statistical properties differ from the remaining it is important to properly select the characteristic function. This must be done so that the separated component would have that singularity.

Let now the medium be such that condition (2.9) is satisfied for functions $\tau_q{}^n$ determined by (1.1), as well as by (2.11). Then instead of (2.9) we have

$$\mu_n^{(i)}(\mathbf{R}_n) = \sum_{\sigma_n}\sum_{q=0}^{n} [\lambda_i{}'' \otimes]^q D_{n-q}^{(\lambda)} \tau_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q}), \quad i = 1 \div N \tag{2.12}$$

A medium for which the central moment can be represented in the form (2.12) is called symmetric. For such medium the use of the superscript is devoid of any meaning.

Taking into account that any of $\mu_n^{(i)}$ can be taken as μ_n, we write

$$\mu_n(\mathbf{R}_n) = \sum_{i=1}^{N} v_i \mu_n^{(i)}(\mathbf{R}_n) = \langle \mu_n(\mathbf{R}_n)\rangle$$

The substitution into that formula of (2.12) yields

$$\mu_n(\mathbf{R}_n) = \sum_{\sigma_n}\sum_{q=0}^{n} D_q^{(\lambda)} D_{n-q}^{(\lambda)} \tau_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q}) \tag{2.13}$$

$$D_q^{(\lambda)} \equiv \sum_{i=1}^{N} v_i [\lambda_i{}'' \otimes]^q$$

Functions $T_{ij\ldots k}^n$, that correspond to (2.13) are of the form

$$T_{ij\ldots k}^n(\mathbf{R}_n) = \sum_{\sigma_n}\sum_{q=0}^{n} D_{ij\ldots l}^{q} D_{m\ldots k}^{n-q} \tau_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q}), \ \tau_2{}^2 = 0 \tag{2.14}$$

It can be shown that $\tau_2{}^2$ is zero by using the condition implied by (2.13) similar to (2.9) imposed on $\tau_q{}^n$.

For $n = 2, 3, 4$ for the tensor field λ_α from (2.13) we have

$$\mu_{\alpha\beta}(\mathbf{R}_2) = D_{\alpha\beta}^{(\lambda)} \tau_0{}^2 (\mathbf{R}_2), \ \ \mu_{\alpha\beta\gamma}(\mathbf{R}_3) = D_{\alpha\beta\gamma}^{(\lambda)} \tau_{0s}^3 (\mathbf{R}_3)$$

$$\tau_{0s}{}^n \equiv \tau_0{}^n + \tau_n{}^n$$

$$\mu_{\alpha\beta\gamma\delta}(\mathbf{R}_4) = D_{\alpha\beta\gamma\delta}^{(\lambda)} \tau_{0s}{}^4(\mathbf{R}_4) + D_{\alpha\beta}^{(\lambda)} D_{\gamma\delta}^{(\lambda)} \tau_{2s}{}^4 (\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) +$$
$$D_{\alpha\gamma}^{(\lambda)} D_{\beta\delta}^{(\lambda)} \tau_{2s}{}^4 (\mathbf{r}_1, \mathbf{r}_3; \mathbf{r}_2, \mathbf{r}_4) + D_{\alpha\delta}^{(\lambda)} D_{\beta\gamma}^{(\lambda)} \tau_{2s} (\mathbf{r}_1, \mathbf{r}_4; \mathbf{r}_2, \mathbf{r}_3)$$

$$\tau_{2s}{}^4 (\mathbf{r}_i, \mathbf{r}_j; \mathbf{r}_k, \mathbf{r}_l) \equiv \tau_2{}^4 (\mathbf{r}_i, \mathbf{r}_j; \mathbf{r}_k, \mathbf{r}_l) + \tau_2{}^4 (\mathbf{r}_k, \mathbf{r}_l; \mathbf{r}_i, \mathbf{r}_j)$$

The explicit form of functions $\tau_{0s}{}^n$ was obtained in [5] in the course of analysis of a random tensor field of the Markovian type. In summarizing the obtained results it is necessary to stress that the invariance (either partial or total) of the medium with respect to inversion (of all or a part) of components leading to (2.13) or (2.12) considerably simplifies the form of central moments and their dependence on concentration and coordinates. (Inversion of components i and j is denoted by $i \leftrightarrow j$ and implies, first, their transposition in space and, second, the substitution $v_i \leftrightarrow v_j$.)

3. Let us consider the case of the two-component medium. Formulas obtained in Sects. 1 and 2 are simplified, since everywhere, where the first component is taken as the separated one, it is necessary to use the substitution $i = j = \ldots = k = 2$, which makes it possible to completely discard these subscripts.

Thus, for instance, from (1.7) we have

$$P^n_{22\ldots 2}(\mathbf{R}_n^{22\ldots 2}) = \langle f_2(\mathbf{r}_1{}^2) f_2(\mathbf{r}_2{}^2) \ldots f_2(\mathbf{r}_n{}^2)\rangle = \langle f(\mathbf{r}_1) f(\mathbf{r}_2) \ldots f(\mathbf{r}_n)\rangle \equiv P_n(\mathbf{R}_n) \tag{3.1}$$

where P_n has the meaning of the probability that all of the n points belong to region U_2 occupied by the second component. Formulas (2.5) assume the form

$$P_n(\mathbf{R}_n) = \sum_{\sigma_n} \sum_{q=0}^{n} v_2{}^q D_{n-q} \pi_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q}) \tag{3.2}$$

$$T_n(\mathbf{R}_n) = \sum_{\sigma_n} \sum_{q=0} (-v_2)^q D_{n-q} \tau_q{}^n (\mathbf{R}_q; \overline{\mathbf{R}}_{n-q})$$

$$D_n = v_2 (1 - v_2)^n + (1 - v_2)(-v_2)^n$$

It follows from this that in the case of a two-dimensional medium the results correspond to the almost symmetric N-component medium. However the determination of central moments by (3.2) yields results that are substantially different from those derived by using (2.10).

In fact, using (2.6) we have

$$\mu_n^{(1)}(\mathbf{R}_n) = T_n^{(1)}(\mathbf{R}_n)[\Delta\lambda \otimes]^n, \quad \Delta\lambda = \lambda_2 - \lambda_1 \tag{3.3}$$

where, as before, the superscript indicates that the first component is taken as the separated one. The singularity of (3.3) is due to that in it the dependence on tensors is separated from the dependence on coordinates. The first of these is determined by tensor $[\Delta\lambda\otimes]^n$ and the second by function $T_n{}^{(1)}(\mathbf{R}_n)$. Thus the form of central moments (3.3) with separated dependence on tensors and coordinates must be considered as specific to two-component media. A formula similar to (3.3) was obtained in the paper (*) in which a mixture of two isotropic components was considered.

*) Moskalenko, V.N. and Maslennikov, S.A., Properties of correlation functions of characteristics of micro-inhomogeneous material. Collection: Problems of Reliability in Structural Mechanics. Theses of proceedings of the All-Union Conference, Vil'nius, 1971.

Let us consider the symmetric medium in more detail. The transition from (1.1) to (2.11) in the case of a two-component medium is effected by the substitution

$$f_1^{(2)}(\mathbf{r}) = 1 - f_2(\mathbf{r}), \quad f_1^{(2)''}(\mathbf{r}) = -f_2''(\mathbf{r}) \tag{3.4}$$

which makes it possible to express functions $P_n^{(2)}$ and $T_n^{(2)}$ determined with the use of $f_1^{(2)}$, in terms of $P_q^{(1)}$ and $T_n^{(1)}$, respectively, as follows:

$$P_n^{(2)}(\mathbf{R}_n) = \sum_{\sigma_n} \sum_{q=0}^{n} (-1)^q P_q^{(1)}(\mathbf{R}_n), \quad T_n^{(2)}(\mathbf{R}_n) = (-1)^n T_n^{(1)} \tag{3.5}$$

Functions $P_q^{(1)}$ and $T_n^{(1)}$ appearing in the right-hand sides of (3.5) are determined according to (3.2), in terms of the second component concentration v_2 and of functions π_q^n and τ_q^n whose properties are determined by the random field $f_2(r)$. Functions $P_n^{(2)}$ and $T_n^{(2)}$ can be expressed in the form (3.2). Then in the case of a symmetric medium functions π_q^n and τ_q^n remain unchanged in virtue of (2.9), i.e. differences in component properties depend only on their concentration, while functions $P_n^{(2)}$ and $T_n^{(2)}$ are determined by formulas (3.2) in which $v_1 = 1 - v_2$ must be substituted for v_2. We denote by $\bar{P}_n^{(1)}$ and $\bar{T}_n^{(1)}$ functions $P_n^{(1)}$ and $T_n^{(1)}$ in which inversion $2 \leftrightarrow 1$ has been effected. We then have

$$P_n^{(2)} = \bar{P}_n^{(1)}, \quad T_n^{(2)} = \bar{T}_n^{(1)} \tag{3.6}$$

which with (3.5) yields

$$\bar{P}_n^{(1)} = \sum_{\sigma_n} \sum_{q=0}^{n} (-1)^q P_q^{(1)}, \quad \bar{T}_n^{(1)} = (-1)^n T_n^{(1)} \tag{3.7}$$

Formulas (3.7) obtained as the corollary of the symmetry condition make it possible to simplify functions P_n and T_n, which eventually results in functions T_n being of the form (2.14).

Using (3.7) we shall show that $\tau_2^2 = 0$. When $n = 2$ we have for function T_n

$$\bar{T}_2^{(1)} = D_2\tau_0^2 + v_1^2\tau_2^2 = D_2\tau_0^2 + v_2^2\tau_2^2 = T_2^{(1)} \tag{3.8}$$

from which, owing to $v_1 \neq v_2$, we obtain the sought result.

It will be seen from (3.8) that the difference between symmetric and arbitrary inhomogeneous media become apparent already in the calculation of functions T_2. Properties of the first case medium are invariant with respect to component inversion $1 \leftrightarrow 2$, hence $\tau_2^2 = 0$, while in the second $\tau_2^2 \neq 0$. Let the second medium be a matrix one. Denoting function T_2 in the first and second cases by T_2^s and T_2^m, respectively, we write

$$T_2^m(\mathbf{R}_2) - T_2^s(\mathbf{R}_2) = v_2^2\tau_2^2(\mathbf{R}_2)$$

Since according to (1.11) and (2.4) function τ_2^2 vanishes when $\mathbf{r}_1 = \mathbf{r}_2$ and $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$, the observed discrepancy can only appear in the intermediate region. For Markovian type media [5] function τ_2^2 is everywhere zero.

A two-component medium considered in [2] satisfied the property

$$P_n^{(2)} = P_n^{(1)}, \quad T_n^{(2)} = T_n^{(1)} \tag{3.9}$$

and was called symmetric. It will be seen that the concentrations of both components is then equal $^1/_2$. Furthermore, in virtue of (3.5) and (3.9) functions P_{2m+1} reduce to linear combinations of functions P_n where $n \leqslant 2m$, and T_{2m+1} vanish. Conditions (3.9) are the limit case of (3.6), since for $v_1 = v_2 = {}^1/_2$ and fulfilment of the symmetry conditions (3.6) converts to (3.9).

The requirement for identity of statistical properties of components of medium (3.9) substantially limits the class of media considered here. Conditions (3.6), on the other hand, represent the extension of the symmetry concept introduced in [2] to a wider class of two-component media, and conditions (2.9) extend it further to media with an arbitrary number of components.

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